Integral representation for ordinary differential equations of Laplace type and exact WKB analysis (Exact steepest descent method)

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数理解析研究所講究録
1168 巻 2000 年 80-92

Integral representation for ordinary differential equations of Laplace type and exact WKB analysis

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1 Introduction

The purpose of this paper is to study the relationship between the exact WKB analysis and integral representations of solutions for the following ordinary differential equations of Laplace type with a large parameter $\eta > 0$:

$$(1) \qquad P\psi = \left(\frac{d^m}{dx^m} + (c_{m-1}x + d_{m-1})\eta \frac{d^{m-1}}{dx^{m-1}} + \cdots + (c_0 x + d_0)\eta^m\right)\psi = 0,$$

where c_j and d_j are complex constants.

In contrast with the case of second-order equations (cf., for example, [V], [CNP], [DP], [KT]), establishment of the exact WKB analysis for higher-order linear ordinary differential equations has not been accomplished yet: To give a complete description of the Stokes geometry (i.e., configuration of Stokes curves) for higher-order equations is still an open problem (cf. [BNR], [AKT1], [AKT2], [A]). However, for equation (1) we have an integral representation of solutions via the Fourier-Laplace transformation and the steepest descent method for such an integral representation enables us to analyze the Borel transform of WKB solutions. In this paper, dealing with the problem by using differential equations, we try to clarify the relationship between the Borel resummed WKB solutions and the integral representation to obtain a characterization of Stokes curves for (1).

Integral representations have been employed by many authors in doing asymptotic analysis of particular examples like Bessel functions (cf. [O] and references cited there). Among others we here refer to the following three works all of which are closely related to our point of view: Uchiyama's study ([U1], [U2]) of several examples of Stokes phenomena, the work of Shudo and Ikeda ([SI1], [SI2]) on chaotic tunneling and Stokes phenomena for some quantum complex dynamics, and hyperasymptotic analysis for integrals due to Berry and Howls ([BH], [H] etc.).

Now the plan of this paper is as follows: We first review the exact WKB analysis for linear ordinary differential equations briefly in §2, and then explain an integral representation of solutions for equation (1) in §3. In §4 we discuss the Airy equation, the simplest example of the Laplace-type equations, to illustrate the relationship between the exact WKB analysis and the integral representation. Finally in §5 and §6 we formulate and prove our main results, which relate the Borel resummed WKB solutions with the integral representation, and consequently obtain a characterization of Stokes curves for (1).

The author would like to express his heartiest thanks to Professor T. Kawai and Professor T. Aoki for their many valuable suggestions and the stimulating discussions with them. This work is supported by Japan Society for the Promotion of Science under Grant-in-Aid for Encouragement of Young Scientists (No. 11740087).

2 WKB solutions & Stokes curves

First of all, let us recall some basic notions and notations used in the exact WKB analysis for equation (1).

A WKB solution of (1) is a solution of the following form:

$$\psi = \exp\left(\int^x \left(\eta S_{-1}(x) + S_0(x) + \eta^{-1} S_1(x) + \cdots\right) dx\right). \tag{2}$$

Such a solution can be readily constructed by substituting (2) into (1). In particular, $S_{-1}(x)$ is a root of the algebraic equation ("characteristic equation")

$$\xi^m + (c_{m-1}x + d_{m-1})\xi^{m-1} + \cdots + (c_0 x + d_0) = 0, \tag{3}$$

while the other coefficients $S_n(x)$ $(n \geq 0)$ are uniquely determined in a recursive manner. Letting $\xi_j(x)$ $(j = 1, \ldots, m)$ denote the roots of (3), we thus find that there exist m WKB solutions ψ_j with the top term $S_{-1}(x) = \xi_j(x)$. In this paper we use $\eta^{-1/2}\psi$ (in place of ψ) as WKB solution of (1) and denote by ψ_j a solution of the form

$$\begin{aligned} \psi_j &= \eta^{-1/2} \exp \int^x \left(\eta S_{-1}(x) + S_0(x) + \eta^{-1} S_1(x) + \cdots\right) dx \\ &= \exp\left(\eta \int^x \xi_j(x) dx\right) \sum_{n=0}^{\infty} \psi_n^{(j)}(x) \eta^{-(1/2+n)}, \end{aligned} \tag{4}$$

as it is more convenient when we discuss its Borel transform and Borel sum below.

WKB solutions are, in general, not convergent. To give an analytic meaning to them, we employ the Borel resummation technique in the exact WKB analysis. That is, for an infinite series $\psi = \exp(\eta y_0(x)) \sum_{n\geq 0} \psi_n(x)\eta^{-(n+\alpha)}$ ($\alpha \notin \mathbb{C} \setminus \mathbb{Z}$) we define its Borel transform (denoted by ψ_B or $\mathcal{B}\psi$ here and in what follows) by

$$\psi_B(x,y) \ \ (\text{ or } \mathcal{B}\psi(x,y)\ ) = \sum_{n=0}^{\infty} \frac{\psi_n(x)}{\Gamma(n+\alpha)} (y+y_0(x))^{n+\alpha-1}, \tag{5}$$

and further, if the Laplace integral

$$\int_{-y_0(x)}^{\infty} e^{-\eta y}\psi_B(x,y)dy \tag{6}$$

(where the path of integration is taken to be parallel with the positive real axis) is well-defined, we regard the interal (6), which is called the Borel sum, as an analytic substitute of ψ.

As is clear from the above definition, when a singular point of the Borel transform meets the path of integration of (6), the Borel sum of a WKB solution is not well-defined and presents the so-called Stokes phenomenon there. To describe where the Borel resummed WKB solutions present Stokes phenomena, we introduce the following terminologies:

Definition 1 (i) A turning point is, by definition, a point where the characteristic equation (3) has a multiple root, i.e., a zero of the discriminant of (3).
(ii) A real 1-dimensional curve (in the x-plane) defined by the following equation is called a Stokes curve:

$$\Im \int_a^x (\xi_j(x) - \xi_{j'}(x))\, dx = 0, \tag{7}$$

where $\xi_j(x)$ and $\xi_{j'}(x)$ are two roots of (3) that merge at a turning point $x = a$.

In the case of second-order equations the whole of Stokes curves thus defined coincides with the set where the Borel resummed WKB solutions present Stokes phenomena. In the case of higher-order equations, applying the local theory near a (simple) turning point developed in [AKT1], we can also find that Stokes phenomena are occurring on a Stokes curve in a sufficiently small neighborhood of a (simple) turning point. However, as is pointed out by Berk et al. ([BNR], cf. [AKT1] also), Stokes phenomena may occur on some other curves different from the above Stokes curves for m-th order equations where $m \geq 3$.

One of the goals of this paper is to give a characterization of the set where the Borel resummed WKB solutions present Stokes phenomena ("true Stokes curves" or "effective Stokes curves") for equation (1) by using the integral representation of solutions.

3 Integral representation

The Fourier-Laplace transformation with a large parameter

$$\hat{\psi}(\xi) = \int \exp(-\eta x\xi)\psi(x)dx, \quad \psi(x) = \int \exp(\eta x\xi)\hat{\psi}(\xi)d\xi \tag{8}$$

transforms equation (1) into the following first-order differential equation:

$$\widehat{P}\hat{\psi} \stackrel{\text{def}}{=} \eta^{m-1}\left(-C(\xi)\frac{\partial}{\partial\xi} - C'(\xi) + \eta D(\xi)\right)\hat{\psi} = 0, \tag{9}$$

where

$$C(\xi) = c_{m-1}\xi^{m-1} + \cdots + c_0, \quad D(\xi) = \xi^m + d_{m-1}\xi^{m-1} + \cdots + d_0. \tag{10}$$

Hence, by solving (9) explicitly, we can readily obtain the following integral representation of solutions of equation (1):

$$\psi(x) = \int e^{\eta(x\xi+g(\xi))}\frac{1}{C(\xi)}d\xi, \tag{11}$$

where $g(\xi)$ is a function satisfying $dg/d\xi = D(\xi)/C(\xi)$.

The integral representation (11) is an integral containing a large parameter η with the phase function

$$f(x,\xi) = x\xi + g(\xi). \tag{12}$$

(Note that x should be regarded just as a parameter here.) Its asymptotic behavior for $\eta \to \infty$ can be analyzed by applying the saddle point method or the steepest descent method to (11). Let us here recall the definition of saddle points and that of steepest descent paths:

Definition 2 (i) A saddle point of (11) (or a saddle point of $f(x,\xi)$) is a non-degenerate critical point of $f(x,\xi)$, i.e., a point where $\partial f/\partial\xi = 0$ and $\partial^2 f/\partial\xi^2 \neq 0$ are satisfied.
(ii) A steepest descent path of (11) (or a steepest descent path of $\Re f(x,\xi)$) is a solution curve of the vector field

$$-\mathrm{grad}_{(\sigma,\tau)}\Re f(x,\xi) = -\left(\frac{\partial(\Re f)}{\partial\sigma}, \frac{\partial(\Re f)}{\partial\tau}\right) \tag{13}$$

(or, equivalently, $(-\partial(\Im f)/\partial\tau, \partial(\Im f)/\partial\sigma)$, the Hamilton vector field of $\Im f(x,\xi)$), where σ and τ respectively denote the real part and the imaginary part of ξ.

Let Σ_{sad} denote the set of saddle points of (11). Since

$$\frac{\partial f}{\partial\xi} = x + \frac{dg}{d\xi} = x + D(\xi)/C(\xi), \tag{14}$$

the roots $\{\xi_j(x)\}_{j=1,\cdots,m}$ of (3) are the saddle points of (11), i.e.,

Lemma 1

$$\Sigma_{\text{sad}} = \{\xi_1(x), \cdots, \xi_m(x)\}. \tag{15}$$

(Here and in what follows we consider the problem in a generic situation, that is, we assume that the roots of $C(\xi) = 0$ and those of $D(\xi) = 0$ are mutually distinct and further that every $\xi_j(x)$ is a simple root of (3).) On the other hand, letting Σ_{sing} denote the set of singular points of the integrand of (11), we find

$$\Sigma_{\text{sing}} = \{\text{zeros of } C(\xi)\} \bigcup \{\infty\}. \tag{16}$$

As we will see later, the steepest descent method for the integral representation (11) is closely related to the exact WKB analysis of equation (1). To illustrate this close relationship, we first discuss the Airy equation in the subsequent section.

4 An example

The Airy equation

$$\left(\frac{d^2}{dx^2} - \eta^2 x\right)\psi = 0 \tag{17}$$

is the simplest example of Laplace-type equations (1). In this case, since $C(\xi) = -1$ and $D(\xi) = \xi^2$, the phase function is given by $f(x,\xi) = x\xi - \xi^3/3$ and the integral representation (11) becomes the following:

$$\psi(x) = \int e^{\eta(x\xi - \xi^3/3)} d\xi. \tag{18}$$

(The insignificant minus sign has been omitted here.) There exist two saddle points: $\xi = \pm x^{1/2}$. In what follows, as there is no essential difference between them, we take up a saddle point $\xi = x^{1/2}$ and consider the integral (18) along the steepest descent path Γ passing through $\xi = x^{1/2}$.

Employing a change of variable defined by $y = -f(x,\xi) = -x\xi + \xi^3/3$, we rewrite the integral (18) along Γ in the following manner (cf. Figure 1):

$$\begin{aligned} \psi(x) &= \int_\Gamma e^{\eta(x\xi - \xi^3/3)} d\xi \\ &= \int_{\tilde{\Gamma}} e^{-\eta y} \left[\left(\frac{dy}{d\xi}\right)^{-1}\Bigg|_{\Gamma_+} - \left(\frac{dy}{d\xi}\right)^{-1}\Bigg|_{\Gamma_-} \right] dy \\ &= \int_{\tilde{\Gamma}} e^{-\eta y} \left[\frac{1}{\xi^2 - x}\Bigg|_{\xi=\xi_+(x,y)} - \frac{1}{\xi^2 - x}\Bigg|_{\xi=\xi_-(x,y)} \right] dy. \end{aligned} \tag{19}$$

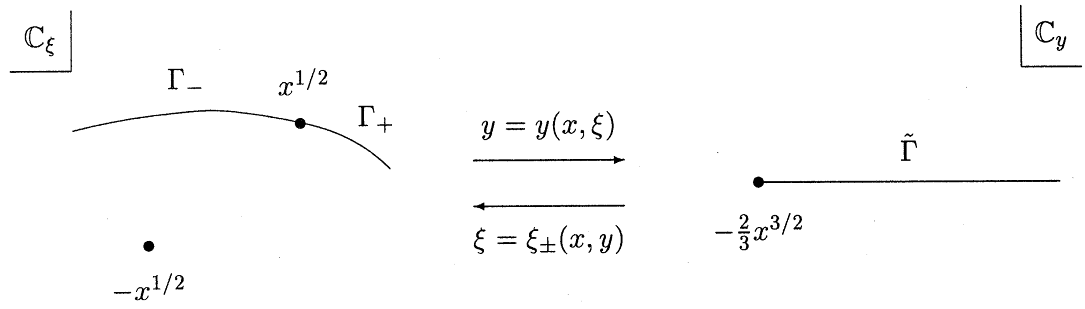

Figure 1

Note that, since $\xi = x^{1/2}$ is a non-degenerate critical point of $f(x,\xi)$, two steepest descent paths (denoted by $\Gamma_\pm$ respectively, cf. Figure 1) emanate from $x^{1/2}$ and the inverse change of $y = -f(x,\xi)$ has two branches $\xi = \xi_\pm(x,y)$ with the following Puiseux expansion there:

$$\xi_\pm(x,y) = x^{1/2} \pm \frac{1}{x^{1/4}}\left(y + \frac{2}{3}x^{3/2}\right)^{1/2} + \cdots. \tag{20}$$

The change of variable $y = -f(x,\xi)$ and its inverse enjoy the following homogeneity: Letting ζ and s denote $\xi/x^{1/2}$ and $(3y)/(4x^{3/2}) + 1/2$ respectively, we find that the inverse change $\zeta_\pm = \zeta_\pm(s)$ corresponding to $\xi_\pm(x,y)$ becomes an algebraic function of one variable s determined by the equation $\zeta^3 - 3\zeta - 4s + 2 = 0$. Having this homogeneity in mind, we denote the following functions by $\varphi_\pm(s)$:

$$\varphi_\pm(s) = \frac{1}{(\zeta_\pm(s))^2 - 1} = \frac{x}{(\xi_\pm(x,y))^2 - x}. \tag{21}$$

It follows from (20) that the difference $\varphi_+ - \varphi_-$ has the Puiseux expansion

$$\varphi_+ - \varphi_- = \frac{\sqrt{3}}{2}s^{-1/2}\chi(s), \quad \chi(s) = 1 + \cdots \tag{22}$$

at $s = 0$. Note that $\chi(s)$ is holomorphic at $s = 0$ (i.e., it does not contain any term with half-integral powers) since $\varphi_+ - \varphi_-$ becomes $-(\varphi_+ - \varphi_-)$ after the analytic continuation along a tiny circle turning around $s = 0$. Now, by a straightforward computations, we can verify that both $\varphi_\pm(s)$ satisfy

$$\left(s(1-s)\frac{d^2}{ds^2} + \left(\frac{3}{2} - 3s\right)\frac{d}{ds} - \frac{8}{9}\right)\varphi_\pm = 0. \tag{23}$$

This is Gauss' hypergeometric differential equation with parameters $(\alpha, \beta, \gamma) = (2/3, 4/3, 3/2)$. Hence, in view of (22) and Kummer's list of solutions of the hypergeometric equation (cf. [BMP, §2.9]), we obtain

$$\varphi_+ - \varphi_- = \frac{\sqrt{3}}{2} s^{-1/2} F\left(\frac{1}{6}, \frac{5}{6}, \frac{1}{2}; s\right), \tag{24}$$

where $F(\alpha, \beta, \gamma; z)$ denotes Gauss' hypergeometric function. That is,

$$\left.\frac{1}{\xi^2 - x}\right|_{\xi=\xi_+(x,y)} - \left.\frac{1}{\xi^2 - x}\right|_{\xi=\xi_-(x,y)} = \left.\frac{\sqrt{3}}{2}\frac{1}{x}\left[s^{-1/2} F\left(\frac{1}{6}, \frac{5}{6}, \frac{1}{2}; s\right)\right]\right|_{s=(3y)/(4x^{3/2})+1/2}. \tag{25}$$

The right-hand side of (25) is (a constant multiple of) the Borel transform of a WKB solution of the Airy equation (17) (cf. [KT, §2.2]). We thus conclude that in the case of the Airy equation the integral representation (19) along the steepest descent path passing through a saddle point gives the Borel sum of a WKB solution.

5 Relationship between exact WKB analysis and steepest descent method for the integral representation

In this section we try to generalize the result for the Airy equation proved in the preceding section and establish the correspondence between the Borel resummed WKB solutions and the integral representation along the steepest descent paths passing through saddle points for the Laplace-type equations (1).

Similarly to the preceding section, we take up a saddle point $\xi = \xi_j(x)$ and employ a change of variable defined by $y = -f(x, \xi) = -(x\xi + g(\xi))$ to rewrite the integral (11) along the steepest descent path $\Gamma^{(j)}$ passing through $\xi = \xi_j(x)$ in the following manner:

$$\begin{aligned} (26)\ \psi(x) &= \int_{\Gamma^{(j)}} e^{\eta(x\xi + g(\xi))} \frac{1}{C(\xi)} d\xi \\ &= \int_{\tilde{\Gamma}^{(j)}} e^{-\eta y} \left[\left.\left(C(\xi) \frac{dy}{d\xi} \right)^{-1} \right|_{\Gamma_+^{(j)}} - \left.\left(C(\xi) \frac{dy}{d\xi} \right)^{-1} \right|_{\Gamma_-^{(j)}} \right] dy \\ &= -\int_{\tilde{\Gamma}^{(j)}} e^{-\eta y} \left[\left.\frac{1}{C(\xi)x + D(\xi)}\right|_{\xi=\xi_+(x,y)} - \left.\frac{1}{C(\xi)x + D(\xi)}\right|_{\xi=\xi_-(x,y)} \right] dy, \end{aligned}$$

where $\tilde{\Gamma}^{(j)}$ is a path emanating from $-y_j(x) \stackrel{\text{def}}{=} -f(x, \xi_j(x))$ and running parallel with the positive real axis. Let us denote the following functions by $\varphi_\pm(x, y)$:

$$\varphi_\pm(x, y) = -\left.\frac{1}{C(\xi)x + D(\xi)}\right|_{\xi=\xi_\pm(x,y)}. \tag{27}$$

As a genaralization of (23), we can verify that both $\varphi_\pm(x,y)$ satisfy some partial differential equation. To make the statement more specific, we introduce the following notations:

$$(28)\qquad P_B = \frac{\partial^m}{\partial x^m} + (c_{m-1}x + d_{m-1})\frac{\partial^m}{\partial x^{m-1}\partial y} + \cdots + (c_0 x + d_0)\frac{\partial^m}{\partial y^m},$$

$$(29)\qquad \widehat{P}_B = \frac{\partial^{m-1}}{\partial y^{m-1}}\left(-C(\xi)\frac{\partial}{\partial \xi} - C'(\xi) + D(\xi)\frac{\partial}{\partial y}\right).$$

Then we have

Proposition 1

$$(30)\qquad P_B\varphi_\pm(x,y) = 0.$$

Proof. As the argument is the same for both $\varphi_\pm$, we omit the suffix $\pm$ in this proof. Using Cauchy's integral formula, we write $\varphi(x,y)$ as

$$(31)\qquad \varphi(x,y) = -\frac{1}{2\pi i}\oint \frac{1}{C(\xi)x + D(\xi)}\bigg|_{\xi=\xi(x,y')}\frac{dy'}{y'-y}.$$

Employment of a change of variable $y' = -(x\xi + g(\xi))$ then tells us that

$$(32)\qquad \varphi(x,y) = -\frac{1}{2\pi i}\oint \frac{1}{C(\xi)(y + x\xi + g(\xi))}d\xi,$$

where the integration is done along a tiny circle around $\xi^\dagger$, the point corresponding to y through the above change of variable.

Letting $\phi(\xi,z)$ denote $(C(\xi)(z+g(\xi)))^{-1}$, we find

$$(33)\qquad \begin{aligned}\widehat{P}_{B,(\xi,z)}\phi(\xi,z) &= \frac{\partial^{m-1}}{\partial z^{m-1}}\left\{-C(\xi)\frac{\partial}{\partial\xi} - C'(\xi) + D(\xi)\frac{\partial}{\partial z}\right\}\left(\frac{1}{C(\xi)(z+g(\xi))}\right)\\ &= \frac{\partial^{m-1}}{\partial z^{m-1}}\left\{-\frac{\partial}{\partial\xi}\left(\frac{1}{z+g(\xi)}\right) + \frac{D(\xi)}{C(\xi)}\frac{\partial}{\partial z}\left(\frac{1}{z+g(\xi)}\right)\right\} = 0.\end{aligned}$$

On the other hand,

$$(34)\qquad \frac{\partial^m}{\partial x^{m-j}\partial y^j}\oint \phi(\xi, y + x\xi)d\xi = \oint \left(\frac{\partial}{\partial y}\right)^m \left(\xi^{m-j}\phi(\xi, y + x\xi')\right)\bigg|_{\xi'=\xi} d\xi,$$

and

$$(35)\qquad x\frac{\partial^m}{\partial x^{m-j}\partial y^j}\oint \phi(\xi, y + x\xi)d\xi = \oint \left(\frac{\partial}{\partial y}\right)^m \left(x\xi^{m-j}\phi(\xi, y + x\xi)\right) d\xi$$

$$= \oint \left(\frac{\partial}{\partial y}\right)^{m-1} \frac{\partial}{\partial \xi'} \left(\xi^{m-j}\phi(\xi, y + x\xi')\right)\Big|_{\xi'=\xi} d\xi$$

$$= \oint \left(\frac{\partial}{\partial y}\right)^{m-1} \left(-\frac{\partial}{\partial \xi}\right) \left(\xi^{m-j}\phi(\xi, y + x\xi')\right)\Big|_{\xi'=\xi} d\xi.$$

Note that, in obtaining the last equality, we have done the integration by parts based on the following relation:

(36)
$$\frac{\partial}{\partial \xi}\left(\xi^{m-j}\phi(\xi, y + x\xi')\right) = \frac{\partial}{\partial \xi'}\left(\xi^{m-j}\phi(\xi, y + x\xi')\right)\Big|_{\xi'=\xi} + \frac{\partial}{\partial \xi}\left(\xi^{m-j}\phi(\xi, y + x\xi')\right)\Big|_{\xi'=\xi}.$$

Using these relations (33)–(35), we obtain

$$P_B\varphi(x, y) = -\frac{1}{2\pi i}\oint \left[\widehat{P}_{B,(\xi,y)}\phi(\xi, y + x\xi')\right]\Big|_{\xi'=\xi} d\xi = 0. \tag{37}$$

This completes the proof. Q.E.D.

As in the preceding section, let us consider the difference $\varphi_+(x, y) - \varphi_-(x, y)$. Thanks to Proposition 1, we can verify that this difference gives the Borel transform of a WKB solution by the following argument:

First, the reference point $-y_j(x)$ corresponding to the saddle point $\xi_j(x)$ in question satisfies

$$\frac{dy_j(x)}{dx} = \frac{d}{dx}\left(x\xi_j(x) + g(\xi_j(x))\right) = \xi_j + x\xi_j' + g'(\xi_j)\xi_j' = \xi_j(x). \tag{38}$$

Hence $y_j(x)$ can be written as $\int^x \xi_j(x)dx$. That is, $y_j(x)$ coincides with the phase factor of the WKB solution ψ_j. In what follows we assume that the phase factor of ψ_j has the same normalization with $y_j(x)$.

Second, $\varphi_+ - \varphi_-$ has the Puiseux expansion

$$\varphi_+ - \varphi_- = \left(y + y_j(x)\right)^{-1/2}\chi(x, y) \tag{39}$$

at $y = -y_j(x)$, where $\chi(x, y)$ contains terms with integral powers only. (This property for $\chi(x, y)$ is proved by the same reasoning as the one used in the preceding section.) In particular, the inverse Borel transform (formal Laplace transform) of $\varphi_+ - \varphi_-$ is an infinite series of the same form with the WKB solution ψ_j:

$$\mathcal{B}^{-1}\left(\varphi_+ - \varphi_-\right) = \exp\left(\eta y_j(x)\right)\sum_{n=0}^{\infty}\chi_n(x)\eta^{-(1/2+n)}. \tag{40}$$

Third, it follows from Proposition 1 that $\mathcal{B}^{-1}(\varphi_+ - \varphi_-)$ satisfies the differential equation (1). This together with the uniqueness of WKB solutions of (1) (i.e., the fact that the coefficients $S_n(x)$ $(n \geq 0)$ are uniquely determined once $S_{-1}(x)$ is fixed) implies that

$$\mathcal{B}^{-1}(\varphi_+ - \varphi_-) = C(\eta)\psi_j \tag{41}$$

holds with some infinite series $C(\eta) = C_0 + \eta^{-1}C_1 + \cdots$ where each C_n is a constant.

We have thus verified that $\varphi_+(x, y) - \varphi_-(x, y)$ is the Borel transform of $C(\eta)\psi_j$. In other words, we have proved the following

Proposition 2 *For Laplace-type equations (1) the integral representation (26) along the steepest descent path $\Gamma^{(j)}$ passing through a saddle point $\xi = \xi_j(x)$ gives the Borel sum of an appropriately normalized WKB solution with the top term $S_{-1}(x) = \xi_j(x)$.*

Note that the singular points of the integrand of the integral representation (11) is described by Σ_{sing} and that the change of variable $y = -f(x, \xi) = -(x\xi + g(\xi))$ employed above becomes singular only at $\Sigma_{\text{sad}} \cup \Sigma_{\text{sing}}$. Among these singular points each point of Σ_{sing} corresponds to the point at infinity in the y-variable (while every point of Σ_{sad} corresponds to some finite point). Hence the singular points of $\varphi_\pm(x, y)$ except $y = \infty$ are described by the image of Σ_{sad} under the change of variable $y = -f(x, \xi)$. As a consequence we obtain the following characterization of "true Stokes curves":

Proposition 3 *If the Borel resummed WKB solutions of Laplace-type equation (1) present Stokes phenomena (that is, x belongs to a "true Stokes curve"), then some steepest descent path of $\Re f(x, \xi)$ connects two saddle points $\xi_j(x)$ and $\xi_{j'}(x)$ of the integral representation. Conversely, if a steepest descent path connects two saddle points, the Borel resummed WKB solutions generically present Stokes phenomena. (For the meaning of "generically" see Remark (i) below.)*

Remark (i) We believe that the "conversely" part of the statement of Proposition 3 is always true for Laplace-type equations, although a delicate cancellation may extinguish a Stokes phenomenon even in such a situation that a steepest descent path connects two saddle points for some integral representations. (For a concrete example of such cancellations see [AKT2, Example 2.1].) In any case, practically speaking, Proposition 3 provides a sufficiently useful characterization of true Stokes curves, as it is not difficult to check if a delicate cancellation occurs for concrete examples.
(ii) Occasionally a steepest descent path emanating from a saddle point $\xi_j(x)$ may return to $\xi_j(x)$ itself. This situation is related to the degeneracy of Stokes geometry (i.e., existence of Stokes curves connecting two turning points).
(iii) Since $1/(C(\xi)x + D(\xi))$ is bounded at each point of Σ_{sing}, the argument of this section also verifies the Borel summability of WKB solutions for Laplace-type equations.

6 Discussion on characterization of Stokes curves

A characterization of true Stokes curves for Laplace-type equations described in Proposition 3 seems quite different from the original definition (7) of Stokes curves. Before ending this paper, we discuss their relationship in this section.

Let us consider the situation that two saddle points $\xi_j(x)$ and $\xi_{j'}(x)$ are located on a level curve of $\Im f(x,\xi)$. This is a slightly weaker situation than that discussed in Proposition 3 as level curves of $\Im f(x,\xi)$ are not necessarily connected. Such a weaker situation is, however, more closely related to the original definition (7) of Stokes curves. As a matter of fact, we can prove the following

Proposition 4 *Let $\xi_j(x)$ and $\xi_{j'}(x)$ be two saddle points of $f(x,\xi)$. Then the following two conditions are equivalent:*
(i)

$$\Im f(x,\xi_j(x)) = \Im f(x,\xi_{j'}(x)). \tag{42}$$

(ii) There exists a closed path γ with the base point x satisfying

$$\xi_j(x) \textit{ becomes } \xi_{j'}(x) \textit{ after the analytic continuation along } \gamma, \tag{43}$$

$$\Im \int_\gamma \xi_j(z)dz = 0. \tag{44}$$

Proof. Let $\mathcal{R}$ denote the Riemann surface of the algebraic equation (3) defined by

$$\begin{aligned} \mathcal{R} &= \{\,(x,\xi)\,;\,\xi^m + (c_{m-1}x + d_{m-1})\xi^{m-1} + \cdots + (c_0x + d_0) = 0\,\} \\ &= \{\,(x,\xi)\,;\,x = x(\xi) \overset{\text{def}}{=} -\frac{D(\xi)}{C(\xi)}\,\}. \end{aligned} \tag{45}$$

Note that $\mathcal{R}$ is a variety in $\mathbb{C}_x \times \mathbb{C}_\xi$ parametrized by ξ (cf. Figure 2). Let $(x,\xi_j(x))$ and $(x,\xi_{j'}(x))$ be two points in $\mathcal{R}$. Then integration by parts on a path (denoted by C) connecting $\xi_j(x)$ and $\xi_{j'}(x)$ verifies

$$\begin{aligned} f(x,\xi_j(x)) - f(x,\xi_{j'}(x)) &= x(\xi_j(x) - \xi_{j'}(x)) + \int_C \frac{D(\xi)}{C(\xi)}d\xi \\ &= \int_C \xi x'(\xi)d\xi \\ &= \int_\gamma \xi(x)dx, \end{aligned} \tag{46}$$

where γ is the projection onto the x-space of $\tilde{C}$, the lift of C to the Riemann surface $\mathcal{R}$. Taking the imaginary part of both sides of (46), we obtain the equivalence in question.

Q.E.D.

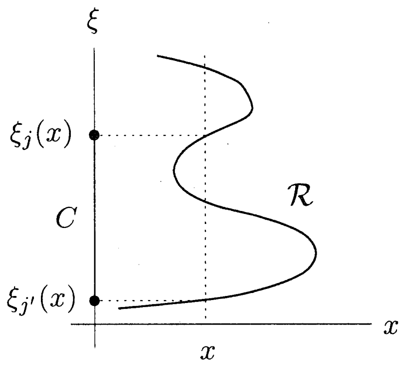

Figure 2

Proposition 4 is a refined version of the discussion done in [AKT3].

The original definition (7) of Stokes curves can be regarded as a special case of the condition (ii) in Proposition 4: If x is a point on the curve defined by (7), then we find that x satisfies the condition (ii) with a closed path γ running along the curve (7) from x to a turning point a and returning to x after making one turn around a. Furthermore the new Stokes curve of the example of Berk et al. ([BNR]) also has an expression both of the form (7) and (44). It is not known, however, whether it is always possible to write the condition (44) in the form of (7).

In this way the definition (7) of Stokes curves is related to the condition (i) in Proposition 4, a slightly weaker condition than that in the statement of Proposition 3. Taking Proposition 3 and this observation into account, we can conclude that the geometry of steepest descent paths is describing the so-called "Riemann sheet structure" of the Borel trasform of WKB solutions of Laplace-type equations (1).

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